

Generally covariant model of a scalar field with high frequency dispersion and the cosmological horizon problem

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Short distance structure of spacetime may show up in the form of high frequency dispersion. Although such dispersion is not locally Lorentz invariant, we show in a scalar field model how it can nevertheless be incorporated into a generally covariant metric theory of gravity provided the locally preferred frame is dynamical. We evaluate the resulting energy-momentum tensor and compute its expectation value for a quantum field in a thermal state. The equation of state differs at high temperatures from the usual one, but not by enough to impact the problems of a hot big bang cosmology. We show that a superluminal dispersion relation can solve the horizon problem via superluminal equilibration, however it cannot do so while remaining outside the Planck regime unless the dispersion relation is artificially chosen to have a rather steep dependence on wavevector.

I. INTRODUCTION

Recent studies of the role of trans-Planckian physics in the Hawking effect and in inflationary cosmology have exploited scalar fields with high frequency dispersion as a model of how short distance physics might affect the behavior of quantum fields in such settings. These models involve a preferred frame in which the distinction between high and low frequency is made. If such a matter field is to be coupled to the spacetime metric in a generally covariant theory of gravity, the preferred frame must be treated as a dynamical quantity rather than as a fixed background structure. One way of doing this was formulated by us in a previous paper [1], in which the preferred frame is determined by a dynamical unit timelike vector field u^a . Using this formulation we obtained an expression for the stress tensor of a scalar field with dispersion.

In the present paper we use this stress tensor to evaluate the equation of state in flat spacetime for a thermal state of the field. This equation of state is then used in a fluid description of the matter field in a cosmological model, and the implications for a hot big bang cosmology are examined. We also investigate under what circumstances superluminal equilibration associated with high

frequency dispersion can solve the cosmological horizon problem.

In considering only the flat space thermal state of the matter field we are adopting an adiabatic approximation which precludes effects related to out of equilibrium phenomena including particle creation in the dynamical background of a cosmological metric such as were considered recently for fields with high frequency dispersion in the context of inflation [2,3]. This is a good approximation for frequencies larger than the expansion rate H . It can therefore be used to study the effects of dispersion for frequencies of order k_0 provided $H \ll k_0$.

II. MODEL FIELD THEORY

Various possibilities exist for the kinetic terms in the action for u^a . For the purposes of illustration in this paper we choose the “minimal theory” of Ref. [1],

$$S_{min}[g_{ab}, u^a, \lambda] = \int d^4x \sqrt{-g} \left(-a_1 R - b_1 F^{ab} F_{ab} + \lambda (g_{ab} u^a u^b - 1) \right), \quad (1)$$

where

$$F_{ab} := 2\nabla_{[a} u_{b]}. \quad (2)$$

The field λ is just a Lagrange multiplier whose variation enforces the constraint that u^a be a unit vector.

For the matter content we are interested in a scalar field with high frequency dispersion $\omega^2 = |\vec{k}|^2 [1 + g(|\vec{k}|/k_0)]$, where g is a function that vanishes at zero, and k_0 is a constant with the dimensions of inverse length which sets the scale for deviations from Lorentz invariance. It has been suggested that such a modified dispersion relation might arise in loop quantum gravity [4,5], or in string theory or other approaches to quantum gravity, or more generally from an unspecified modification of the short distance structure of spacetime (see for example [6,7]). Possible observational consequences have been the subject of recent study (see for example [6,8–10] and references therein), and the role of such dispersion in the Hawking process [7] and in the generation of inflationary primordial density fluctuations [2,3] have been examined.

Absent a reliable theory of such modifications, it makes sense simply to expand in k/k_0 . We consider here the lowest order modification for a scalar field that is invariant under rotations and analytic in k , which is given by

$$\omega(k)^2 = |\vec{k}|^2 - |\vec{k}|^4/k_0^2, \quad (3)$$

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where k_0 is a dimensionful parameter. With $k_0^2 > 0$ this yields a subluminal group velocity, and with $k_0^2 < 0$ it is superluminal. The pathology at $|\vec{k}| = k_0$ in the subluminal case is irrelevant since (3) is only regarded as the first terms in an expansion. When we evaluate the equation of state, we shall impose a cutoff that avoids this pathology.

The dispersion relation (3) can be produced by adding a term to the action with four spatial derivatives. Previously this has been done in 1+1 dimensional models (for a review see [7]), and recently such models have been generalized to field theory in a background 3+1 dimensional Robertson-Walker spacetime [2,3]. Here we extend these models to a general 3+1 dimensional spacetime. This can be accomplished in a generally covariant manner, consistently with spatial rotation invariance, with the action $S_\varphi = \int d^4x \sqrt{-g} \mathcal{L}_\varphi$, where

$$\mathcal{L}_\varphi = \frac{1}{2} \left(\nabla^a \varphi \nabla_a \varphi + k_0^{-2} (D^2 \varphi)^2 \right). \quad (4)$$

Here D^2 is the covariant spatial Laplacian, i.e.,

$$D^2 \varphi = -D^a D_a \varphi = -q^{ac} \nabla_a (q_b{}^c \nabla_b \varphi), \quad (5)$$

where D_a is the spatial covariant derivative operator [11] and q_{ab} is the (positive definite) spatial metric orthogonal to the dynamical unit vector u^a ,

$$q_{ab} := -g_{ab} + u_a u_b. \quad (6)$$

The equation of motion for the metric takes the Einstein form

$$G_{ab} = 8\pi G \left(T_{ab}^{(u)} + T_{ab}^{(\varphi)} \right), \quad (7)$$

where $G = 1/16\pi a_1$, and with

$$T_{ab}^{(u)} = -4b_1 (F_{am} F_b{}^m - \frac{1}{4} F^2 g_{ab}) + 2\lambda u_a u_b, \quad (8)$$

$$\begin{aligned} T_{ab}^{(\varphi)} &= \nabla_a \varphi \nabla_b \varphi - \mathcal{L}_\varphi g_{ab} \\ &- k_0^{-2} \left[2D^2 \varphi u^m u_{(a} \nabla_{|m|} D_{b)} \varphi + 2\nabla_m (D^2 \varphi q_{(a}{}^m \nabla_{b)} \varphi \right. \\ &\quad \left. - \nabla^m (q_{ab} D^2 \varphi D_m \varphi) \right]. \end{aligned} \quad (9)$$

(The constraint equation $g_{ab} u^a u^b = 1$ has been used to drop the contribution to (8) that would have come from the variation of $\sqrt{-g}$ in the constraint term of the action (1).)

The equation of motion for the field u^a takes the form

$$\nabla^b F_{ba} = -\frac{1}{2b_1} \left(\lambda u_a + \frac{1}{2} \frac{\delta S_\varphi}{\delta u^a} \right). \quad (10)$$

with

$$\begin{aligned} \frac{\delta S_\varphi}{\delta u^a} &= 2k_0^{-2} u^b \\ &\left[D^2 \varphi \nabla_{(a} (q_b) {}^m \nabla_m \varphi) - \nabla_m (D^2 \varphi q_{(a}{}^m \nabla_{b)} \varphi) \right]. \end{aligned} \quad (11)$$

Contracting (10) with u^a we obtain an expression for λ in terms of the other fields:

$$\lambda = 2b_1 u^a \nabla^b F_{ab} - \frac{1}{2} u^a \frac{\delta S_\varphi}{\delta u^a}, \quad (12)$$

where, from (11),

$$u^a \frac{\delta S_\varphi}{\delta u^a} = 4k_0^{-2} u^a u^b D^2 \varphi \nabla_{[m} \varphi \nabla_{a]} q_b{}^m. \quad (13)$$

III. THERMAL STATE IN A ROBERTSON-WALKER COSMOLOGY

Let us now specialize to a Robertson-Walker (RW) spacetime, in the semiclassical framework where the metric and u^a are treated as classical fields and the scalar field is a quantum field (which therefore has a well-defined thermal equilibrium state). In the field equations for the metric and for u^a we take the expectation value of the φ -terms. Assuming u^a shares the RW symmetry it must be the cosmological rest frame. The tensor F_{ab} then vanishes, so λ is just determined by the matter term in (12). (If there are further terms involving u^a in the action then there are additional contributions to λ .)

With RW symmetry we have $\nabla_a u_b = H q_{ab}$, where $H = \dot{a}/a$ is the usual Hubble “constant”. Iterating this identity we find $\nabla_a q_{mn} = H(q_{am} u_n + q_{an} u_m)$, from which it follows that the contraction (13) is given by

$$u^a \frac{\delta S}{\delta u^a} = 6k_0^{-2} H \dot{\varphi} D^2 \varphi, \quad (14)$$

where $\dot{\varphi} = u^m \nabla_m \varphi$.

Suppose now that the scalar field is well approximated by an adiabatically evolving thermal state. This would be the case if (i) there are interactions that produce an equilibration rate which is large compared to the expansion rate, and (ii) the thermal frequency is also large compared to the expansion rate. In this case the expectation value of (the Hermitian part of) the operator $\dot{\varphi} D^2 \varphi$ vanishes, since the operator is odd under time reversal while a thermal state is invariant. Using (12) and (14) this implies that, in the minimal model, $\langle \lambda \rangle = 0$. (If other terms are included in the action for u^a then $\langle \lambda \rangle$ would not vanish, although it would still not receive contributions from this matter field.) Therefore in this model $\langle T_{ab}^{(u)} \rangle = 0$, so the only contribution to the cosmological stress tensor comes from the scalar field.

A. Thermal equation of state

Consider now the expectation value $\langle T_{ab}^{(\varphi)} \rangle$ of the stress tensor (9)—or more precisely of its Hermitian part—in a thermal state. Note first that time reversal invariance of

the thermal state requires an even number of time derivatives of φ , and spatial isotropy requires an even number of spatial derivatives of φ , in order for the thermal expectation value not to vanish. Thus terms with an odd number of derivatives of φ do not contribute. Let us call this the “odd derivative rule”. We can use this rule to see that the expectation value $\langle \mathcal{L}_\varphi \rangle$ vanishes. Integrating by parts, \mathcal{L}_φ can be expressed as a term that vanishes since φ satisfies its equation of motion, plus the total derivative of an expression involving only terms with an odd number of derivatives of φ . The total derivative can be taken out of the expectation value, hence that term vanishes.

The part of (9) multiplied by k_0^{-2} has three terms. The expectation value of the first term has a single time derivative, hence vanishes by time reversal symmetry. The expectation value of the third term is the gradient of an expression with three spatial derivatives which vanishes by the odd derivative rule. The second term can be integrated by parts, and the resulting total derivative piece has vanishing expectation value by the odd derivative rule, which leaves only $2q_{(a}^m \langle D^2 \varphi \nabla_m \nabla_{b)} \varphi \rangle$. In a thermal state this must have the form $A u_a u_b + B q_{ab}$, where u^a is the rest frame defined by the thermal bath. To find A we contract with $u^a u^b$, which yields $A = 0$ due to the factor q_a^m . To find B we contract with q^{ab} and divide by 3, hence this term contributes $-\frac{2}{3} \langle D^2 \varphi D^2 \varphi \rangle$. The expectation value of the stress tensor thus takes the form

$$\langle T_{ab} \rangle = \rho u_a u_b + P q_{ab}, \quad (15)$$

with energy density ρ and pressure P given by

$$\rho = \langle \dot{\varphi}^2 \rangle \quad (16)$$

$$P = \langle \frac{1}{3} (D\varphi)^2 - \frac{2}{3} k_0^{-2} (D^2 \varphi)^2 \rangle. \quad (17)$$

To evaluate the density and pressure we expand the field in Fourier components using the dispersion relation (3) and sum the contributions from the modes, weighting each by the thermal expectation value of the number operator $e^{\omega(\vec{k})/T} - 1$, which yields

$$\rho = \int \frac{d^3 k}{2\pi^3 \omega(\vec{k})} \frac{\omega(\vec{k})^2}{e^{\omega(\vec{k})/T} - 1}, \quad (18)$$

$$P = \frac{1}{3} \int \frac{d^3 k}{2\pi^3 \omega(\vec{k})} \frac{\omega(\vec{k})^2 - |\vec{k}|^4/k_0^2}{e^{\omega(\vec{k})/T} - 1}. \quad (19)$$

For low temperatures $T \ll k_0$, only modes with $k \ll k_0$ contribute significantly, hence we recover the standard result for massless radiation, $P = \frac{1}{3}\rho$, and the energy density scales as $\rho \propto T^4$.

Before looking at the exact temperature dependence, let us consider the high temperature limit. The nature of this limit depends on whether the dispersion is sub- or super-luminal. In the superluminal case $k_0^2 < 0$, we can sensibly use the dispersion relation (3) out to arbitrarily

large k , so in the high temperature limit only wave vectors $|\vec{k}| \gg |k_0|$ are relevant. For such wave vectors we have $\omega(\vec{k})^2 \simeq |\vec{k}|^4/|k_0|^2$, hence the energy density scales as $\rho \propto T^{5/2} k_0^{3/2}$, and $P \simeq \frac{2}{3}\rho$. This should be taken only as a qualitative indication of what might occur, since one would expect that for $|\vec{k}| \gtrsim k_0$ further terms in the k -expansion of the dispersion relation become important.

In the subluminal case $k_0^2 > 0$ we must impose a cutoff so as not to enter the unphysical region where the dispersion relation would yield $\omega(\vec{k})^2 < 0$. We choose to impose the cutoff at $|\vec{k}| = k_0/\sqrt{2}$, where $\omega(\vec{k})$ attains its maximum value. This is similar to a lattice cutoff for which, in one dimension, the dispersion relation is $\omega(k) = (2/\delta) \sin k\delta/2$, and for which $|k| \leq \pi/\delta$ exhausts all the independent modes. With this cutoff in place, $\omega(\vec{k})/T \ll 1$ for all modes in the high temperature limit, hence the expectation value of the number operator tends toward $T/\omega(\vec{k})$. Thus the energy density scales as $\rho \propto T k_0^3$ in this limit. As for the equation of state, the ratio of the integrals yields $P/\rho \simeq 0.174$.

The interpolation between the low and high temperature limits can be determined by a numerical calculation of the integrals. In Fig. 1 we plot P/ρ vs. $\log_{10}(T/k_0)$. It is seen that the equation of state smoothly connects the low and high temperature limits. For the subluminal case most of the interpolation takes place over the range of temperatures $10^{-1.5} < T/k_0 < 10^{-0.5}$, while in the superluminal case most of the interpolation takes place over the somewhat higher range $10^{-1} < T/k_0 < 10$.

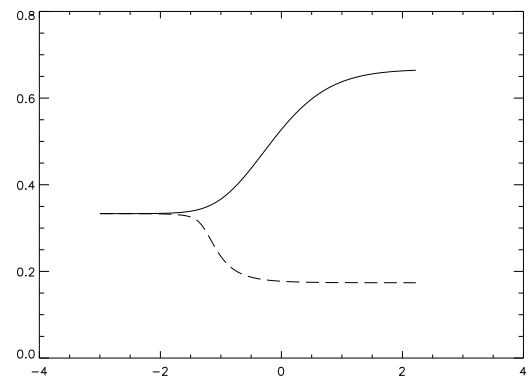


FIG. 1. Thermal expectation values for P/ρ vs. $\log(T/k_0)$ in the presence of high frequency dispersion. The dashed and solid curves are for the sub- and superluminal cases respectively.

B. Cosmological implications

The modified equation of state derived here can be used in a cosmological model where thermal matter with non-Lorentz-invariant dispersion acts as a source of gravity. The Bianchi identity implies the energy-momentum

tensor is divergence-free, which for a perfect fluid implies $\dot{\rho} = -3H(\rho + p)$. Together with the equation of state this determines the evolution of the temperature. In the superluminal case local energy-momentum conservation is also independently implied by the matter field equations as usual, however in the subluminal case the dynamics is not self-contained because of the high wave vector cutoff. If the cutoff is imposed at a fixed proper wave vector, new modes are added as the universe expands, and the field dynamics does not specify into which state these modes are born. The assumptions of a thermal stress tensor and of energy-momentum conservation require that, whatever their birth state, they rapidly equilibrate with the rest of the modes. This is reasonable under the assumption that the system is in the adiabatic regime.

It is interesting to ask whether high frequency dispersion could have any impact on the cosmological problems that led to the invention of the inflationary scenario. In particular, could it provide a solution of these problems not requiring inflation? The modification of the equation of state caused by the dispersion would affect the quantitative details of the evolution of the scale factor at times when the typical wavevectors are of order k_0 or greater, which is presumably only in the very early universe. However, as we have seen, the equation of state only changes from $P = \rho/3$ to $P = (2/3)\rho$ in the superluminal case and to $P = 0.17\rho$ in the subluminal case. Neither are significant enough to qualitatively change the dynamics or horizon size.

C. Superluminal equilibration and the horizon problem

In the superluminal case, the fact that influences travel faster than light at wavevectors $k \gtrsim k_0$ opens up the possibility of solving the horizon problem via superluminal equilibration. The coordinate distance covered by a wavepacket with proper group velocity v_g is $\Delta x = \int v_g dt/a$. For a dispersion relation that goes as $\omega \sim k^n$ at large wave vectors the group velocity goes as $v_g \sim k^{n-1}$. It is easy to show that the typical wavevector at the peak of a thermal distribution scales as a^{-1} (as long as the dispersion relation is homogeneous), hence the typical group velocity scales as a^{1-n} , so we have $\Delta x \sim \int dt/a^n$. If this diverges then the horizon problem is solved assuming the framework of this model.

With the above dispersion relation, the equation of state is $P = (n/3)\rho$, which according to the Einstein equation yields the evolution of the scale factor $a(t) \propto t^{2/(n+3)}$. Thus $\Delta x \sim \int dt t^{-2n/(n+3)}$, which diverges at the lower limit for any $n \geq 3$.

Unfortunately it does not make much sense to view this as a solution to the horizon problem, because most of the Δx is traversed during a regime in which the typical wavevector and energy density are so much larger than the Planck scale that we have no reason to trust

the semiclassical model at all. In particular, we have checked that in order for Δx to surpass the horizon size, the evolution must be extrapolated all the way back to a time at which the typical wavevector is roughly k_0 times $[10^{57}(k_0/k_{Planck})^2]^{1/(n-3)}$. Presuming that k_0 is within a few orders of magnitude of the Planck scale, this typical wave vector exceeds the Planck scale unless $n \gtrsim 50$. Since we have no theory that would determine the maximum exponent n appearing in a superluminal dispersion relation, it would seem artificial at this stage to adjust n in order to achieve a sub-Planckian resolution of the horizon problem.

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